hep-th/0203061
PUPT-2025

A Note on Fluxes and Superpotentials In $\mathcal{M}$-theory Compactifications On Manifolds of G_2 Holonomy

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We consider the breaking of $\mathcal{N}=1$ supersymmetry by non-zero G-flux when $\mathcal{M}$-theory is compactified on a smooth manifold X of G_2 holonomy. Gukov has proposed a superpotential W to describe this breaking in the low-energy effective theory. We check this proposal by comparing the bosonic potential implied by W with the corresponding potential deduced from the eleven-dimensional supergravity action. One interesting aspect of this check is that, though W depends explicitly only on G-flux supported on X, W also describes the breaking of supersymmetry by G-flux transverse to X.

March 2002

1. Introduction

One route to possible $\mathcal{M}$-theory phenomenology is to consider $\mathcal{M}$-theory compactifications on eleven-dimensional spaces of the form $M_4 \times X$, where M_4 denotes flat Minkowski space. When the seven-fold X possesses a metric of G_2 holonomy, then $M_4 \times X$ is a vacuum solution of Einstein's equation. Further, there exists one covariantly constant spinor on X, leading to an effective theory with $\mathcal{N} = 1$ supersymmetry in four dimensions. However, in contrast to $\mathcal{M}$-theory compactifications on Calabi-Yau four-folds [1], if we generalize this background ansatz to allow for non-zero G-flux and a warped product metric on $M_4 \times X$, then no supersymmetric vacua away from the trivial $G = 0$ background exist. This was demonstrated, for instance, in [2].

The issue of supersymmetry-breaking by G-flux on $M_4 \times X$ is interesting, as this G-flux also generates a cosmological constant in the four-dimensional theory [3], [4]. As in the case of compactifications on Calabi-Yau four-folds [5], the breaking of supersymmetry by G-flux can be effectively described in the four-dimensional theory by introducing a superpotential W [6] for the moduli of the compactification.

In the case of compactifications on Calabi-Yau four-folds, the superpotentials proposed in [5] have been directly verified by a Kaluza-Klein reduction of the effective $\mathcal{M}$-theory action [7]. One purpose of this note is to perform a similar check of the superpotential describing G-flux in compactifications on manifolds of G_2 holonomy. We compare the bosonic potential derived from W with the corresponding bosonic potential obtained from the $\mathcal{M}$-theory effective action. In addition we want to explore a phenomenon that arises in $\mathcal{M}$-theory compactification to four dimensions but not in compactification to three dimensions on a Calabi-Yau four-fold. As in the Freund-Rubin solution, while preserving the four-dimensional symmetries, the G-field can have a component with all indices tangent to M_4, possibly triggering the breaking of supersymmetry. We will show that the minimal superpotential that describes the components of G along X actually also incorporates the component tangent to M_4. This observation has interesting implications for the structure of the parameter space of compactifications with G-flux.

The outline for this note is the following. In Section 2 we review the low-energy structure of Kaluza-Klein compactification of the $\mathcal{M}$-theory effective action on a smooth manifold[1] X of G_2 holonomy. We directly find the effective bosonic potential for G.

[1] Compactification on a smooth X is not phenomenologically viable, since the low-energy theory will contain only abelian vector multiplets with no charged matter. When X is allowed to have appropriate singularities, non-abelian gauge-groups and charged chiral matter [8], [9] can be present, but of course the low-energy supergravity approximation is no longer valid.

In Section 3 we introduce and motivate the superpotential W. We then derive the bosonic potential for G following from W. We find a potential which naively differs from the result of Section 2.

Finally in Section 4, we show how the two can be reconciled.

2. Kaluza-Klein Reduction of $\mathcal{M}$-Theory on Manifolds of G_2 Holonomy

In this section, we first review the structure of the massless $\mathcal{N}=1$ multiplets that arise when eleven-dimensional supergravity is compactified on X (with $G=0$), as has been discussed in [10], [11], [12]. The four-dimensional effective theory possesses $b_2(X)$ abelian vector superfields V^j and $b_3(X)$ neutral chiral superfields Z^i. These superfields describe massless modes of the flat C-field and the metric on X.

To describe these modes explicitly, let us choose bases of harmonic forms $\{\omega_j\}$ for $\mathcal{H}^2(X)$ and $\{\phi_i\}$ for $\mathcal{H}^3(X)$. We then make a Kaluza-Klein ansatz for C,

$$C = c^i(x)\,\phi_i + A^j_\mu(x)\,dx^\mu{\wedge}\omega_j\,. \tag{2.1}$$

This ansatz is slightly oversimplified as it applies only when the G-flux is trivial. The scalars c^i and the vectors A^j_μ describe the holonomies of a flat C-field. Because these holonomies take values in $U(1)$ and not $\mathbb{R}$, the fields appearing in (2.1) (in particular the scalars c^i) should also be regarded as taking values in $U(1)$ rather than $\mathbb{R}$. This observation deserves emphasis—it can also be understood by noting that under "large" gauge-transformations which add to C a closed 3-form on X of appropriately normalized periods, the c^i undergo integral shifts.

We also note that when X has G_2 holonomy, $b_1(X)=0$, for the same reasons as in the case of Calabi-Yau three-folds. So no harmonic 1-forms on X appear in the ansatz for C. Each vector $A^j_\mu(x)$ in (2.1) gives rise to one abelian vector superfield V^j, and each scalar $c^i(x)$ in (2.1) appears as the real component of the complex scalar z^i in the chiral superfield Z^i.

The corresponding imaginary components of the z^i describe massless fluctuations in the background metric on X. Recall that associated to any metric of G_2 holonomy on X is a unique covariantly constant (hence closed and co-closed) 3-form Φ. Given any such metric, we may associate to it the cohomology class $[\Phi]$ in $H^3(X;\mathbb{R})$, and this assignment is invariant under diffeomorphisms of X. In fact, as was shown by Joyce [13], the moduli space of G_2 holonomy metrics on X, modulo diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $b_3(X)$. Further, near a point in the moduli space corresponding to the equivalence class of metrics associated to Φ, the moduli space is locally diffeomorphic to an open ball about $[\Phi]$ in $H^3(X;\mathbb{R})$. These results imply that massless modes of the metric on X may be parametrized by introducing $b_3(X)$ scalars $s^i(x)$ defined by

$$\Phi = s^i(x)\,\phi_i\,. \tag{2.2}$$

(The s^i are presumed to fluctuate around some point away from the origin.) Thus the s^i in (2.2) naturally combine with the c^i as $z^i = c^i + i\,s^i$. Note that this holomorphic combination $C + i\Phi$ on X is analogous to the complexified Kähler class $B + iJ$ familiar from compactification on Calabi-Yau three-folds.

We now recall the bosonic action of eleven-dimensional supergravity [14],

$$S_{11} = \frac{1}{2\kappa_{11}^2}\int d^{11}x\Big[\sqrt{-g}R - \frac{1}{2}G\wedge\star G - \frac{1}{6}C\wedge G\wedge G\Big]\,. \tag{2.3}$$

Higher derivative corrections in the $\mathcal{M}$-theory effective action, such as the $CI_8(R)$ term [15], will not be relevant for the following. We also find it useful to introduce T_2 and T_5, the $M2$-brane and $M5$-brane tensions, and to recall the relations between κ_{11}^2, T_2, and T_5 (as derived, for example, in [16])

$$\begin{aligned}\frac{1}{2\kappa_{11}^2} &= \frac{1}{2\pi}T_2T_5\,,\\ T_5 &= \frac{1}{2\pi}T_2^2\,.\end{aligned} \tag{2.4}$$

Henceforth, we set $T_2 = 1$ to obtain the standard flux quantization conditions on G. In these units, $\kappa_{11}^2 = 2\pi^2$ and $T_5 = 1/2\pi$.

We wish to determine the potential induced for the moduli Z^i when $G \neq 0$. We assume that G respects the Lorentz symmetry of M_4 and so decomposes as

$$G = G_0 + G_X\,, \tag{2.5}$$

where $G_0 = G|_{M_4}$ and $G_X = G|_X$. As explained in [17], Dirac quantization on X generally requires that $\frac{1}{2\pi}G_X - \frac{1}{2}\lambda$ has integral periods, where $\lambda = p_1(X)/2$. So if λ were not even in $H^4(X;\mathbb{Z})$, our earlier assumption that $G_X = 0$ would not have been consistent quantum mechanically. However, the following simple argument (see footnote 2 of [11]) implies that when X is a spin seven-fold (one consequence of G_2 holonomy), then λ is always even. Consider $S^1\times X$. This is a spin eight-fold, and $p_1(S^1\times X) = p_1(X)$, so it suffices to consider λ on $S^1 \times X$. But on any spin eight-fold, λ being even is equivalent to the intersection form on H^4 being even (for a proof of this standard fact via index theory, see [17]). Finally, the intersection form on $S^1 \times X$ is even for trivial reasons. So we learn that $\frac{1}{2\pi} G_X$ must have integral periods on X, consistent with $G_X = 0$.

The quadratic GG term in S_{11} now descends directly to a pair of terms in the low-energy action[2] for the metric moduli,

$$\delta S_4^{(GG)} = -\frac{1}{8\pi^2} \int d^4x \left[\mathrm{vol}(X)\, G_0 \wedge \star G_0 + \sqrt{-g_4} \int_X G_X \wedge \star G_X \right] . \tag{2.6}$$

A term in the low-energy action for the moduli of the C-field is also induced from the CGG Chern-Simons term in S_{11},

$$\delta S_4^{(CGG)} = -\frac{1}{8\pi^2} \int d^4x \left[G_0 \int_X C \wedge G_X \right] . \tag{2.7}$$

(In computing this interaction, which will be converted to an ordinary potential in Section 4, we have dropped boundary terms from infinity on M_4 which can be absorbed in the background value of C. A factor of three has arisen because the Chern-Simons term is cubic in C.)

Let us now adopt a slightly more suggestive notation. We dualize the flux G_0 on M_4 by introducing a scalar f satisfying

$$G_0 = f\, dx^0 \wedge \ldots \wedge dx^3 \,, \quad \star G_0 = -f \,, \tag{2.8}$$

in a coframe adapted to the metric. We also define

$$\theta \equiv \frac{1}{4\pi} \int_X C \wedge G_X \,. \tag{2.9}$$

The expression θ is a seven-dimensional Chern-Simons form on X. It is not well-defined as a real number, due to the fact that C is only defined up to shifts $C \to C + \phi_i$, where the harmonic forms $\frac{1}{2\pi}\phi_i$ are normalized to have integral periods on X. At first glance, one might have thought that θ is consequently defined only modulo $\pi \cdot$ integer. In fact, because the class λ of X is even, a careful treatment of θ, as given in Section 3 of [18], shows that θ is actually well-defined modulo $2\pi \cdot$ integer. Hence our notation is correct in suggesting that θ is an angle.

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[2] We will not distinguish notationally between the four-, seven-, and eleven-dimensional Hodge $\star$, but the distinction should be clear from context.

The four-dimensional effective potential for C and Φ is then determined from (2.6) and (2.7) after we pass to Einstein frame, rescaling the four-dimensional metric $g_{\mu\nu} \to 2\pi^2\,\mathrm{vol}(X)^{-1}\,g_{\mu\nu}$. We find this potential to be

$$V(C,\Phi) = -\frac{1}{32\pi^6}\,\mathrm{vol}(X)^3\,f^2 + \frac{\pi^2}{2}\,\mathrm{vol}(X)^{-2}\int_X G_X\wedge\star G_X + \frac{\theta}{2\pi} f\,. \tag{2.10}$$

The additional factors of $\mathrm{vol}(X)$ in the f^2 term arise from the explicit factor in (2.6) and the scaling of the four-dimensional Hodge $\star$. This term also has the "wrong" sign as it is really a kinetic energy, so we slightly abuse the terminology in referring to V as a "potential". The θ term, as it descends from the eleven-dimensional Chern-Simons term, remains independent of $\mathrm{vol}(X)$ under the rescaling.

3. The Superpotential

3.1. Motivating the Superpotential

We can now introduce the superpotential $W(Z^i)$, essentially proposed in [6], which describes the breaking of supersymmetry by G-flux on X. We consider

$$W(Z^i) = \frac{1}{8\pi^2}\int_X \left(\frac{1}{2}C + i\,\Phi\right)\wedge G_X\,. \tag{3.1}$$

The relative factor of $1/2$ between the two terms in W is required by supersymmetry. For under a variation

$$C \to C + \delta C, \quad \Phi \to \Phi + \delta\Phi\,, \tag{3.2}$$

the superpotential varies as

$$W \to W + \frac{1}{8\pi^2}\int_X (\delta C + i\,\delta\Phi)\wedge G_X\,. \tag{3.3}$$

Note that a relative factor of 2 has appeared in the variation of the first term due to its quadratic dependence on C and the fact that, whereas $d\Phi = 0$, $dC = G$. δW is linear in $\delta C + i\,\delta\Phi$ as required for holomorphy. The condition for unbroken supersymmetry and zero cosmological constant of a four-dimensional $\mathcal{N}=1$ supergravity theory with superpotential W is that, in the vacuum,

$$W = dW = 0\,. \tag{3.4}$$

The latter condition, for W above, is sufficient to imply that G_X must vanish in a supersymmetric vacuum with zero cosmological constant.

A discussion of the proper interpretation of the former condition is called for, since the term $\int_X C\wedge G_X/(4\pi)^2 = \theta/4\pi$ in W is only well-defined modulo $1/2\cdot$integer. We have no way to pick a natural definition of this expression as a real number, so the best we can do is to say that all possibilities differing by $\theta \to \theta + 2\pi$ are allowed. Thus, the theory depends on an integer that is not fixed when the C-field on X (and its curvature G_X) are given.

What is the physical interpretation of this integer? Heuristically, it corresponds to the value of the period $\int_X G_7/(2\pi)^2$ ($= T_5 \int_X G_7/2\pi$ in our conventions), where G_7 is the seven-form field dual to G. In fact, the classical equation of motion $dG_7 = -\frac{1}{2}G\wedge G + \ldots$ (the $\ldots$ being gravitational corrections that we do not consider here) shows that $\int_X G_7/(2\pi)^2$ is not constant because G_7 is not closed. As observed by Page [19], what is constant and should be quantized is rather $\int_X (G_7 + \frac{1}{2}C\wedge G)/(2\pi)^2$, and since $\frac{1}{2}\int_X C\wedge G/(2\pi)^2$ ($= \theta/2\pi$) is anyway only defined mod an integer, we introduce no additional ambiguity if we take $\int_X (G_7 + \frac{1}{2}C\wedge G) = 0$. Thus, $\frac{1}{2}\int_X C\wedge G/(2\pi)^2$ can "stand in" for the G_7-flux, and the possibility of adding an integer to its value amounts to the possibility of shifting the G_7-flux by an integer number of quanta.

At this point, we can see that the parameter space of M-theory compactifications with G-flux is not, as one might have supposed, a product of the space of C-fields on X with a copy of $\mathbb{Z}$ parametrized by the G_7-flux. Rather, the parameter space is fibered over the space of C-fields, with the fiber being a copy of $\mathbb{Z}$; but the fibration is non-trivial. Consider varying C by $C \to C + C'$, where $dC' = 0$. If C' has trivial periods, this change in the C-field is topologically trivial. But $\frac{1}{2}\int_X C\wedge G/(2\pi)^2$ changes in a non-trivial way. If $G/2\pi$, restricted to X, is divisible by an integer m, then the change in $\frac{1}{2}\int_X C\wedge G/(2\pi)^2$ will always be an integer multiple of m, and so the change in the G_7-flux is likewise a multiple of m. Thus, the only invariant under this process is the value of G_7 modulo m. For example, if $m = 1$, G_7 can be varied arbitrarily, and the overall parameter space of C-fields plus G_7-flux is connected. Only when $G_X = 0$ is the parameter space what one would expect naively: a product of the space of C-fields with a copy of $\mathbb{Z}$ parameterizing the G_7-flux.

For precise computations, it is awkward to work with G_7, since (as G is closed and G_7 is not) the theory is naturally formulated with a three-form field C and not with a dual six-form field. In Section 4, by treating the C-field quantum mechanically, we will show how the picture we have just described is reproduced if one works with G, rather than (as in the last few paragraphs) the dual G_7. The need to work quantum mechanically when one formulates the discussion in terms of G should not come as a surprise; duality typically relates a classical description in one variable to a dual quantum mechanical description.

Previously, it was suggested [2] that a superpotential describing the effects of G-flux along M_4 and X would be

$$W = \int_X G_7 + \int_X (C + i\Phi)\wedge G_X \,. \tag{3.5}$$

One might have naively thought that the two terms in (3.5) involving G_7 and G_X were describing independent effects due to G-flux along M_4 and X. As we have seen, this interpretation would be problematic because generically the parameter space of G_7-flux fibers non-trivially over the space of C-fields on X. A closely related observation is that neither the term involving G_7 nor the term involving G_X individually respects holomorphy. So the relative normalization of the terms is not arbitrary but fixed by holomorphy, contrary to what the naive interpretation would suggest. In fact, taking $\int_X (G_7 + \frac{1}{2} C\wedge G) = 0$, we see that (3.5) corresponds, up to normalization, to the proposed superpotential in (3.1), whose holomorphy we verified and in which G_7 does not explicitly appear.

The superpotential (3.1) can also be motivated and its normalization fixed from an argument given in [5]. Consider an $M5$-brane having worldvolume $\mathbb{R}^{2,1} \times \Sigma$, where Σ is a 3-cycle on X which is calibrated by Φ. Such a calibrated 3-cycle is a supersymmetric 3-cycle [20] and has minimal volume, $\mathrm{vol}(\Sigma) = |\int_\Sigma \Phi|$, within its homology class. So the wrapped $M5$-brane appears as a BPS domain wall in the four-dimensional theory. The tension τ of such a BPS domain wall in the low-energy $d = 4$, $\mathcal{N} = 1$ Wess-Zumino model describing the Z^i is [21]

$$\tau = 2|\Delta W| \,, \tag{3.6}$$

where ΔW is the change in W upon crossing the wall.

On the other hand, the $M5$-brane is a magnetic source for G_X, and the class of G_X must change upon crossing the wall. This change is $\Delta G_X = 2\pi\, \delta_\Sigma$, where δ_Σ is the fundamental class of Σ. Hence (assuming $C = 0$) we have

$$\begin{aligned} 2|\Delta W| &= \frac{1}{(2\pi)^2} \left| \int_X \Phi\wedge\Delta G_X \right| \,, \\ &= \frac{1}{2\pi} \left| \int_\Sigma \Phi \right| = T_5 \mathrm{vol}(\Sigma) \,, \end{aligned} \tag{3.7}$$

as we expect for an M5-brane domain wall. The dependence on C follows from supersymmetry.

We now derive the the bosonic potential U which follows from the superpotential W. Recall that, in terms of the Kähler potential $\mathcal{K}$, the metric on moduli space $g_{i\bar{j}}$, and W, the potential U is given by [22]

$$U = \exp(\mathcal{K})\Big[g^{i\bar{j}} D_i W \overline{D_j W} - 3|W|^2\Big]\,, \tag{3.8}$$

where $D_i W = \partial_i W + \partial_i \mathcal{K} \cdot W$ is the covariant derivative of W as a section of a line bundle over the moduli space parametrized locally by the Z^i.

To evaluate U, we need to know $g_{i\bar{j}}$ and $\mathcal{K}$. The metric $g_{i\bar{j}}$ can be determined most directly from the kinetic terms of the c^i fields. These kinetic terms arise from reducing the term $-\frac{1}{8\pi^2}\int G\wedge\star G$ in the action S_{11}. We find these kinetic terms to be (in Einstein frame)

$$\mathcal{L}_{kin} = -\frac{1}{4}\mathrm{vol}(X)^{-1}\,\partial_\mu c^i \partial^\mu c^j \int_X \phi_i\wedge\star\phi_j\,. \tag{3.9}$$

On the other hand, the metric $g_{i\bar{j}}$ appears in the four-dimensional action in the term $\mathcal{L}_{kin} = -g_{i\bar{j}}\partial_\mu z^i \partial^\mu \overline{z}^j$. So (3.9) determines the metric $g_{i\bar{j}}$ to be

$$g_{i\bar{j}} = \overline{\partial_j}\partial_i\mathcal{K} = \frac{1}{4}\mathrm{vol}(X)^{-1}\int_X \phi_i\wedge\star\phi_j\,. \tag{3.10}$$

We now claim that the Kähler potential is given, up to shifts $\mathcal{K}(Z^i, \overline{Z^i}) \to \mathcal{K} + f(Z^i) + f^*(\overline{Z^i})$, by

$$\mathcal{K} = -3\log\left[\frac{1}{2\pi^2}\cdot\frac{1}{7}\int_X \Phi\wedge\star\Phi\right]. \tag{3.11}$$

The general form for $\mathcal{K}$ has appeared in [11], [12], but we must be careful to check the factor of -3 appearing in the normalization of $\mathcal{K}$. We will make this check directly by computing $\overline{\partial_j}\partial_i\mathcal{K}$.

3.2. Mathematical Preliminaries and Computing $\overline{\partial_j}\partial_i\mathcal{K}$

In order that the following be self-contained, we must review a few facts about the group G_2 and metrics of G_2 holonomy (for which a good general reference is [23]). As very lucidly described in [24], the group G_2 can be defined as the subgroup of $GL(7,\mathbb{R})$ preserving a particular 3-form on $\mathbb{R}^7$. In terms of coordinates $(x^1,\ldots,x^7)$ on $\mathbb{R}^7$, this 3-form is

$$\Phi = \theta^{123} + \theta^{145} + \theta^{167} + \theta^{246} - \theta^{257} - \theta^{347} - \theta^{356}\,, \tag{3.12}$$

where we abbreviate $\theta^{i_1\cdots i_n} = dx^{i_1}\wedge\ldots\wedge dx^{i_n}$. G_2 also preserves the Euclidean metric $ds^2 = (dx^1)^2 + \ldots + (dx^7)^2$ (as we expect, since G_2 occurs as the holonomy of X and so must be a subgroup of $O(7)$) and hence the dual 4-form with respect to this metric,

$$\star\Phi = \theta^{4567} + \theta^{2367} + \theta^{2345} + \theta^{1357} - \theta^{1346} - \theta^{1256} - \theta^{1247}\,. \tag{3.13}$$

We note that

$$dx^1\wedge\ldots\wedge dx^7 = \frac{1}{7}\Phi\wedge\star\Phi\,, \tag{3.14}$$

so that G_2 also preserves the orientation of $\mathbb{R}^7$.

When X possesses a metric of G_2 holonomy, then at each point p of X there exists a local frame in which the covariantly constant 3-form Φ takes the form in (3.12) and the metric takes the Euclidean form. Hence the local relation (3.14) immediately implies that

$$\frac{1}{7}\int_X \Phi\wedge\star\Phi = \mathrm{vol}(X)\,. \tag{3.15}$$

This relation explains the factor of $\frac{1}{7}$ appearing in $\mathcal{K}$, which, besides being very natural, will give the correct normalization of U.

Because we are primarily interested in 3-forms on X, our interest naturally lies in the G_2 representation $\Lambda^3(\mathbb{R}^7)^*$ consisting of rank-three anti-symmetric tensors. $\Lambda^3(\mathbb{R}^7)^*$ decomposes under G_2 into irreducible representations $\mathbf{1}\oplus\mathbf{7}\oplus\mathbf{27}$. The trivial representation $\mathbf{1}$ is generated by the invariant 3-form (3.12) which we used to define G_2. The $\mathbf{7}$ arises from the fundamental representation of G_2 as a subgroup of $GL(7,\mathbb{R})$ via the map $(\mathbb{R}^7)^* \hookrightarrow \Lambda^3(\mathbb{R}^7)^*$, sending $\alpha \mapsto \star(\alpha\wedge\Phi)$. Note that $\star(\cdot\wedge\Phi)$ defines a (non-zero) G_2-equivariant map, which must be an isomorphism onto its image by Schur's lemma. The $\mathbf{27}$ can then be characterized as the set of those elements λ in $\Lambda^3(\mathbb{R}^7)^*$ which satisfy $\lambda\wedge\Phi = \lambda\wedge\star\Phi = 0$. This identification follows again from the fact that $\cdot\wedge\star\Phi$ and $\cdot\wedge\Phi$ are G_2-equivariant maps.

Now, as is familiar from the case of Calabi-Yau three-folds, the decomposition of $\Lambda^3(\mathbb{R}^7)^*$ into irreducible representations of G_2 implies a corresponding decomposition of Λ^3T^*X under the holonomy. The Laplacian on X respects this decomposition, implying a corresponding classification of the harmonic 3-forms on X, $\mathcal{H}^3(X) \cong \mathcal{H}^3_1(X)\oplus\mathcal{H}^3_7(X)\oplus\mathcal{H}^3_{27}(X)$. In fact, the Laplacian of X as an operator on p-forms depends only on the representation of the holonomy, not on p. Hence the dimension of $\mathcal{H}^p_R(X)$ for some representation R depends only on R, not p. Thus, since $\mathcal{H}^1(X) = \mathcal{H}^1_7(X) = 0$, we also have that $\mathcal{H}^3_7(X) = 0$. Further, from our characterization of the $\mathbf{27}$ above, we see that $\mathcal{H}^3_1(X)$ is orthogonal to $\mathcal{H}^3_{27}(X)$ in the usual inner product $(\alpha,\beta) = \int_X \alpha\wedge\star\beta$.

We are now prepared to evaluate $\overline{\partial_j}\partial_i\mathcal{K}$. First we observe that $\int_X \Phi\wedge\star\Phi$ is a homogeneous function of the s^i of degree $\frac{7}{3}$. This observation follows from (3.15) and the fact that under a scaling of the local coframe $dx^i \to \lambda dx^i$, Φ scales as $\Phi \to \lambda^3\Phi$, or equivalently the s^i scale as $s^i \to \lambda^3 s^i$, and $\mathrm{vol}(X)$ scales as $\mathrm{vol}(X) \to \lambda^7\mathrm{vol}(X)$. Remembering that $s^i = \mathrm{Im}(z^i)$, homogeneity of $\int_X \Phi\wedge\star\Phi$ in the s^i then implies that

$$\partial\mathcal{K}/\partial z^i = i\frac{7}{2}\frac{\int_X \phi_i\wedge\star\Phi}{\int_X \Phi\wedge\star\Phi}\,. \tag{3.16}$$

To evaluate a second derivative of $\mathcal{K}$, we must evaluate $\frac{\partial}{\partial s^i}(\star\Phi)$ arising from the numerator of (3.16). Here $\Phi : s^i \mapsto s^i\phi_i$ is a linear map of the (local) coordinates s^i on the moduli space, and for a fixed metric on X, the Hodge $\star$ is certainly a linear operator on $\mathcal{H}^3(X)$. However, since the operator $\star$ depends on the metric, hence the moduli s^i, in general $\star\Phi(s^i)$ depends nonlinearly on the s^i. Following Joyce [13] (where this derivative appears), we will denote $\Theta(s^i) \equiv \star\Phi(s^i)$ to emphasize this nonlinearity.

Let us restrict to a local coordinate patch (diffeomorphic to $\mathbb{R}^7$) on X and on this patch consider Θ as a map on a small open ball about the canonical 3-form Φ in $\Lambda^3T^*\mathbb{R}^7$. Θ is well-defined, since for Ξ sufficiently small, a local change of frame can always be found[3] taking $\Phi+\Xi$ to the canonical form (3.12). The metric associated to $\Phi+\Xi$ in the frame for which $\Phi+\Xi$ is canonical is the Euclidean metric, and so Θ is very easy to evaluate in this frame.

We now consider the derivative $D\Theta$. $D\Theta$ is locally linear (i.e. linear over $C^\infty(\mathbb{R}^7)$), so it suffices to consider $D\Theta$ as a linear map on $\Lambda^3(\mathbb{R}^7)^*$. Observe that, since Φ is G_2-invariant, $D\Theta$ is actually a G_2-equivariant map. If we denote by π^1, π^7, and π^{27} the projections on $\Lambda^3(\mathbb{R}^7)^* \cong \mathbf{1}\oplus\mathbf{7}\oplus\mathbf{27}$, then Schur's lemma implies that $D\Theta$ decomposes as

$$D\Theta = D\Theta\circ\pi^1 + D\Theta\circ\pi^7 + D\Theta\circ\pi^{27} = a\star\circ\pi^1 + b\star\circ\pi^7 + c\star\circ\pi^{27}\,, \tag{3.17}$$

for some constants a, b, c.

The above expression for $D\Theta$ certainly holds when we consider evaluating $D\Theta$ on the restrictions of elements of $\mathcal{H}^3(X)$ to the local patch. But the expression is also a sensible global expression on X, so it must in fact hold globally. It remains only to evaluate the constants a and c (since $\mathcal{H}^3_7 = 0$, b is irrelevant for us[4]).

[3] This statement follows from a simple dimension count, noting that $\dim GL(7,\mathbb{R}) = 49$, $\dim G_2 = 14$ and $\dim\Lambda^3(\mathbb{R}^7)^* = 35$, so that $\dim GL(7,\mathbb{R}) - \dim G_2 = \dim\Lambda^3(\mathbb{R}^7)^*$.

[4] $b = 1$ for the curious.

We determine a and c by explicit computation in $\Lambda^3(\mathbb{R}^7)^*$. We fix a by once more considering the scaling $dx^i \to \lambda dx^i$, under which $\Phi \to \lambda^3\Phi$ and $\star\Phi \to \lambda^4\star\Phi$. So $a = 4/3$. To fix c, let us consider the 3-form $\Xi = \theta^{123} - \theta^{145}$. Since $\Xi\wedge\Phi = \Xi\wedge\star\Phi = 0$, we see that Ξ transforms in the **27** of G_2. By a change in the frame $dx^2, dx^3 \to (1-\epsilon/2)dx^2, dx^3$ and $dx^4, dx^5 \to (1+\epsilon/2)dx^4, dx^5$, the sum $\Phi + \epsilon\,\Xi$ can be brought to the canonical form, to linear order in ϵ. Dualizing and transforming back to the original frame, we easily see that $\Theta(\Phi + \epsilon\,\Xi) = \star\Phi - \epsilon\star\Xi + \mathcal{O}(\epsilon^2)$, which fixes $c = -1$. So we finally conclude that

$$\frac{\partial}{\partial s^i}(\star\Phi) = \frac{4}{3}\star\pi^1(\phi_i) - \star\pi^{27}(\phi_i)\,. \tag{3.18}$$

This derivative in hand, we evaluate $\partial\mathcal{K}/\partial\overline{z}^j\partial z^i$ as

$$\begin{aligned}\overline{\partial_j}\partial_i\mathcal{K} &= -\frac{7}{4}\frac{1}{\int_X \Phi\wedge\star\Phi}\left\{\int_X \phi_i\wedge\star[\frac{4}{3}\pi^1(\phi_j) - \pi^{27}(\phi_j)] - \frac{1}{\int_X\Phi\wedge\star\Phi}\int_X \phi_i\wedge\star\Phi\cdot\frac{7}{3}\int_X\phi_j\wedge\star\Phi\right\}\,,\\ &= -\frac{1}{4}\mathrm{vol}(X)^{-1}\left\{\frac{4}{3}\int_X \pi^1(\phi_i)\wedge\star\pi^1(\phi_j) - \int_X \pi^{27}(\phi_i)\wedge\star\pi^{27}(\phi_j) - \frac{7}{3}\int_X\pi^1(\phi_i)\wedge\star\pi^1(\phi_j)\right\}\,,\\ &= \frac{1}{4}\mathrm{vol}(X)^{-1}\int_X \phi_i\wedge\star\phi_j\,,\end{aligned} \tag{3.19}$$

where we have noted that $\pi^1(\phi_i) = \Phi\cdot(\int_X\phi_i\wedge\star\Phi)/(\int_X\Phi\wedge\star\Phi)$. Comparing to (3.10), we see that $\mathcal{K}$ is the properly normalized Kähler potential.

3.3. Computing the Potential

The rest of the calculation of U from (3.8) is now direct. We have

$$D_iW = \frac{1}{8\pi^2}\int_X \phi_i\wedge G_X + i\frac{7}{16\pi^2}\left[\frac{\int_X\phi_i\wedge\star\Phi}{\int_X\Phi\wedge\star\Phi}\right]\cdot\int_X(\frac{1}{2}C + i\,\Phi)\wedge G_X\,. \tag{3.20}$$

So

$$\begin{aligned}g^{i\overline{j}}D_iW\overline{D_jW} = \frac{1}{7(2\pi)^4}\int_X\Phi\wedge\star\Phi\cdot\Big\{\int_X G_X\wedge\star G_X &+ \frac{21}{4}\left(\int_X\Phi\wedge\star\Phi\right)^{-1}\cdot\left(\int_X\Phi\wedge G_X\right)^2\\ &+\left(\frac{7}{2}\right)^2\left(\int_X\Phi\wedge\star\Phi\right)^{-1}\cdot\left(\frac{1}{2}\int_X C\wedge G_X\right)^2\Big\}.\end{aligned} \tag{3.21}$$

Here we have noted, for instance, that $g^{i\overline{j}}\int_X\phi_i\wedge G_X\cdot\int_X\phi_j\wedge G_X = \frac{4}{7}\int_X\Phi\wedge\star\Phi\cdot\int_X G_X\wedge\star G_X$. This relation may be checked by expanding $G_X = f^i(s)\star\phi_i$ for some functions $f^i(s)$ (although G_X is independent of the metric, the basis $\{\star\phi_i\}$ is not). So we find

$$
\begin{aligned}
U &= \frac{7^3\pi^2}{2}\left(\int_X \Phi\wedge\star\Phi\right)^{-2}\left[\frac{1}{7}\int_X G_X\wedge\star G_X + \left(\int_X \Phi\wedge\star\Phi\right)^{-1}\left(\frac{1}{2}\int_X C\wedge G_X\right)^2\right], \\
&= \frac{\pi^2}{2}\mathrm{vol}(X)^{-2}\left[\int_X G_X\wedge\star G_X + \mathrm{vol}(X)^{-1}\left(\frac{1}{2}\int_X C\wedge G_X\right)^2\right].
\end{aligned}
\tag{3.22}
$$

Comparing V in (2.10) to U in (3.22), we see that the superpotential produces the correct G_X^2 term, but of course the terms in V involving f do not appear in U. In terms of the angle θ, we can write

$$
U = \frac{\pi^2}{2}\mathrm{vol}(X)^{-2}\int_X G_X\wedge\star G_X + 8\pi^6\,\mathrm{vol}(X)^{-3}\left(\frac{\theta}{2\pi}\right)^2. \tag{3.23}
$$

4. Comparing the Potentials

There seems to be an obvious disagreement between V and U. They do not even depend on the same variables. The component $G_0 = f\,dx^0\wedge\ldots\wedge dx^3$ of the G-field appears in V as

$$
V(f) = -\frac{1}{32\pi^6}\,\mathrm{vol}(X)^3\,f^2 + \frac{\theta}{2\pi}\,f\,. \tag{4.1}
$$

Hence V depends on G_0 as well as on the C-field along X. By contrast, U depends only on the C-field along X.

We will now see that V and U can be reconciled by treating G_0 quantum mechanically. In fact, quantum mechanically, there are only discrete allowed values for f; upon setting it equal to an allowed value, V coincides with U.

A four-form field in 3+1 dimensions is analogous to a two-form field – the curvature of an abelian gauge field – in 1+1 dimensions. We will simply interpret (4.1) as the action for a four-form field and treat it quantum mechanically. (The underlying eleven-dimensional supergravity action also contains couplings of G_0 to fermions, but these are inessential for our present purposes.) In the analogy with 1+1-dimensional abelian gauge theory, the term in (4.1) that is proportional to θ is analogous to the theta-angle of abelian gauge theory.

Classically, if one were allowed to just minimize V with respect to f, having nonzero θ would induce a non-zero value for f and hence a nonzero energy density. This is roughly what emerges from a proper quantum mechanical treatment [25]. If one proceeds naively, the classical value for f in (4.1) is

$$f = 8\pi^5 \, \mathrm{vol}(X)^{-3} \, \theta \, , \tag{4.2}$$

leading to the induced energy density

$$E(\theta) = 8\pi^6 \, \mathrm{vol}(X)^{-3} \left(\frac{\theta}{2\pi}\right)^2 . \tag{4.3}$$

Quantum mechanically, one really wants the energies of all of the states; one finds [25] that the quantum states of this system are labeled by an integer n, with the energy density of the n^{th} state being

$$E_n(\theta) = 8\pi^6 \, \mathrm{vol}(X)^{-3} \left(n + \frac{\theta}{2\pi}\right)^2 . \tag{4.4}$$

A different way to describe the above results is that (4.3) is, indeed, the correct formula for the energy, but in interpreting $\theta/2\pi$ as a real number, one must include all possibilities, differing by the possible addition of an integer. Of course, one could, for any given θ pick the – generically unique – integer that minimizes the energy. However, we prefer to show that our two potentials V and U agree (when V is treated quantum mechanically) without imposing any such restriction. For this, we simply use (4.3), but accepting all real lifts of θ differing by multiples of 2π, as we have anyway done throughout this paper.

Comparing to (3.23), we see that the vacuum energy density $E(\theta)$ due to G_0 is exactly the second term in U, so that the superpotential W indeed captures the effects of supersymmetry-breaking by both G_X and G_0.

Finally, we note that the potential U is positive-definite. In the regime of large $\mathrm{vol}(X)$, for which our effective $\mathcal{M}$-theory action is valid, no vacuum exists and we see a runaway to infinite volume, a familiar situation in supergravity compactifications. We might also consider contributions to the potential from membrane instantons wrapping calibrated 3-cycles Σ on X, but the leading instanton contribution to the potential [11] in the large volume regime is of order $e^{-\mathrm{vol}(\Sigma)}$ and so will not help to stabilize the runaway.

Acknowledgements

The work of C.B. is supported by an NSF Graduate Fellowship and under NSF Grant PHY-9802484. The work of E.W. is supported in part by NSF Grant PHY-0070928.

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